Super-Resolution From Unregistered and Totally Aliased Signals Using Subspace Methods

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Abstract

◆ Limited sampling frequency
  – Physical characteristics of the components
    • Pixel pitch
    • Rate of the A/D converter

◆ Multichannel sampling method
  – Reconstructing a higher resolution signal
    • Using information in aliasing parts
    • Nonlinear problem
      – Offsets and signal coefficient
    • Unique solution
      – $MN \geq L + M - 1$
Introduction

◆ Proposed method in paper
  – Signal reconstruction in super resolution
    • Aliasing components that contain valuable high frequency information
    • Multiple aliased sampled signals with relative offsets

◆ Aliasing in the sampled signals
  – Undersampling of a signal
    • Disadvantage
      – Nuisance and avoided problem
        » Artificial low frequency patterns
        » Jagged edge
      – Not easy to register
        » Subpixel motion estimation
• Advantage
  – Recovering a the high frequency information
    » Higher resolving power than original images

Fig. 1. Examples of aliasing in images. Artificial low-frequency patterns (a) and jagged edges (b) appear due to undersampling.
Fig. 2. Value of aliasing. If an antialiasing lowpass filter is applied to images prior to sampling (a), no additional high frequency information can be recovered by combining multiple images. One can as well interpolate a single image (b). If no filter is applied, the captured images are aliased (c), and high frequency information can be recovered by combining multiple images (d). However, new image registration methods are required to obtain sufficiently high precision.
Classification of sampling methods

- Uniform sampling
  - Sampled periodically at constant time intervals
  - Shannon-Nyquist sampling
- Nonuniform sampling
  - General sampling
    - Known sampling instants
    - Unknown sampling locations
  - Multichannel sampling
    - Known offsets
    - Unknown offsets
    » Uniform sets of samples
    » Nonuniform offset
Fig. 3. Classification of sampling methods. Sampling methods can be divided into uniform and nonuniform methods. The nonuniform sampling methods can be subdivided depend on weather the location are known and the samples are grouped in uniform sets with only unknown offsets. In this paper, we discuss multichannel sampling methods with unknown offsets.
Mathematical Description

- **Mathematical framework**
  - Periodic bandlimited signal with period 1
    \[ f(t) = \sum_{l=-K}^{K} \alpha_l e^{j2\pi lt} \]  
    where \( \alpha_l \) are the \( L=2K+1 \) Fourier coefficients of \( f(t) \).
  - Uniformly sampling at a rate \( N \) over the period \([0,1]\)
    \[ T_0 = \left( \begin{array}{cccc} 0 & 1/N & 2/N & \cdots & \frac{N-1}{N} \end{array} \right) \]
    where \( T_0 \) are the sampling times.
    \[ y_0(n) = f\left( \frac{n}{N} \right) = \sum_{l=-K}^{K} \alpha_l W^{ln} \]  
    where \( W = e^{j2\pi/N} \)
More compact form of eq. (3)

Vector and matrices

\[ y_0 = \begin{pmatrix}
1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
W^{-K} & \cdots & W^{-1} & 1 & W & \cdots & W^K \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
W^{-(N-1)K} & \cdots & W^{-(N-1)} & 1 & W^{N-1} & \cdots & W^{(N-1)K}
\end{pmatrix}
\begin{pmatrix}
\alpha_{-K} \\
\vdots \\
\alpha_{-1} \\
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{-K}
\end{pmatrix} = F^* \alpha \quad (4)\]

where \( y_0 \) is the \( N \times 1 \) sample vector.
\( \alpha \) is the \( L \times 1 \) vector of unknown Fourier coefficients.
and \( F^* \) is the \( N \times L \) inverse discrete Fourier transform (IDFT) matrix.
– Inverse discrete Fourier transform matrix
  • Hermitian transpose of the forward DFT matrix
  • Repeating some of the columns
    – Undersampling
    – Underdetermined

– Another set of sampling time
  • Shifted by an unknown offset $t_m$

$$T_m = \begin{pmatrix} t_m & \frac{1}{N} + t_m & \frac{2}{N} + t_m & \cdots & \frac{N-1}{N} + t_m \end{pmatrix}$$ (5)

where the offsets are relative to the first set of samples, we have $t_0 = 0$. 
– Samples from the mth set

\[ y_m(n) = f \left( \frac{n}{N} + t_m \right) = \sum_{l=-K}^{K} \alpha_l e^{j2\pi l(n/N + t_m)} \]

\[ = \sum_{l=-K}^{K} \alpha_l W_{ln} W_l^t \]

where \( W_{tm} = e^{j2\pi t_m} \)

– Matrix notation

\[ y_m = F^* D_{tm} \alpha = \Phi_{tm} \alpha \]  

(7)

where \( F^* \) is the \( N \times L \) IDFT matrix.

\( D_{tm} \) is an \( L \times L \) diagonal matrix with element \( D_{tm}(l,l) = W_{tm}^l \) \((-K \leq l \leq K)\),

and \( \Phi_{tm} = F^* D_{tm} \).
– Combined into a single vector $y$

$$y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{M-1} \end{pmatrix} = \begin{pmatrix} F^* \\ F^*D_{t_1} \\ \vdots \\ F^*D_{t_{M-1}} \end{pmatrix} \alpha = \begin{pmatrix} \Phi_0 \\ \Phi_{t_1} \\ \vdots \\ \Phi_{t_{M-1}} \end{pmatrix} \alpha = \Phi t \alpha \quad (8)$$

where the basis matrices $\Phi_{t_0}$ are combined into $\Phi t$, with $t = (t_0 \ t_1 \ \ldots \ t_{M-1})$ denoting the offset vector.

– Condition of determined problem

• Known $t$

$$MN > L$$

• Unknown offset

$$MN > L + M - 1 \quad (9)$$
– Summary of the important variable

\( N \)  the number of samples in each set \( y_m \);
\( y_m \)  the length \( N \) vector of the \( m \)th set of samples;
\( L \)  the number of unknown Fourier coefficients \( (L = 2K + 1) \);
\( \alpha \)  the length \( L \) vector of the expansion coefficients \( \alpha_i \) to be reconstructed;
\( M \)  the number of sets of samples;
\( t \)  the length \( M \) vector of the offsets \( t_m \) between the different sets of samples;

- Unknown variable
  - Only fourier coefficient and offset value
- Requiring  the number of fourier coefficient \( (L) \)
  - Estimation of \( L \) at least
Fig. 4. Illustration of the different variables with $M=2$ and a Fourier basis. (a) Time-domain representation of the signal $f(t)$ and its sets of samples $y_0$ (-) and $y_1$ (--). (b) Frequency-domain representation of the absolute values of the signal spectrum (-) and its aliased copies after sampling (--).
– DFT of a set of samples \( y_m \)

\[
Y_m = \frac{1}{N} F_N^* y_m = \frac{1}{N} F_N F^* D_{tm} \alpha
\]  

(10)

where \( F_N \) is a square \( N \times N \) DFT matrix.
\( F^* \) is the \( N \times L \) IDFT matrix.
\( Y_m \) has length \( N \) and is an aliasing and phase shifted version of \( \alpha \).

• Example for \( L=3N \)

\[
Y_m = \frac{1}{N} F_N F^* D_{tm} \alpha = \frac{1}{N} F_N \left( F_N^* F_N^* F_N^* \right) D_{tm} \alpha
\]

\[
= (I \ I \ I) D_{tm} \alpha = \sum_{i=-1}^{1} W^* t_m D'_{tm} i \alpha_i
\]  

(11)

where \( D'_{tm} \) is the \( N \times N \) central part of the \( N \times L \) matrix \( D_{tm} \).
\( \alpha_i \) is the \( i \)th block of length \( N \) from the vector \( \alpha \).
– Example 2.1

- \( M=2, N=2 \), and \( L=3 \)
- Using eq. (8)

\[
y = \begin{pmatrix}
y_0(0) \\
y_0(1) \\
y_1(0) \\
y_1(1)
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 \\
-1 & 1 & -1 \\
W_{t_1}^{-1} & 1 & W_{t_1} \\
-W_{t_1}^{-1} & 1 & -W_{t_1}
\end{pmatrix}\begin{pmatrix}
\alpha_{-1} \\
\alpha_0 \\
\alpha_1
\end{pmatrix} = \Phi_t \alpha
\]

(11)

- Decomposed to diagonal matrix
  - Using eq. (7)

\[
\Phi_t = \begin{pmatrix}
W_{t_1}^{-1} & 1 & W_{t_1} \\
-W_{t_1}^{-1} & 1 & -W_{t_1}
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 \\
-1 & 1 & -1
\end{pmatrix}\begin{pmatrix}
W_{t_1}^{-1} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & W_{t_1}
\end{pmatrix} = F^*D_{t_1}
\]

(12)

\[\rightarrow \text{The offsets and signal coefficients are mixed up, which makes the problem challenging.}\]
Fig. 5. Example of the problem setup. Bandlimited function with two sets of two samples and unknown offset between the sets.
Existence of a unique solution

- Unique mapping from each set of samples to a single space generated by the column of $\Phi_t$

Property of aliasing signals

- Repeating some of the columns $F^*$
- Only multiplication factor $W_{tm}^l$
  
  • Corresponding to overlapping frequencies in the sampled spectrum

\[
\begin{align*}
(\phi_l^l)^T &= \left( f_l^T \ W_{tm}^l f_l^T \cdots \ W_{tm}^{lM-1} f_l^T \right) \\
\text{where}& \quad \phi_l^l \text{ is the } l\text{th column of the matrix.} \\
\text{and } & \quad W_{tm}^l \text{ is are the multiplication factors.}
\end{align*}
\]
Example 3.1

- $M=2, L=5,$ and $N=4$
- Existing two overlapping frequency components
  - First and fifth column of the IDFT matrix
  - Same basis vector

\[
\Phi_t = \begin{pmatrix} \Phi_0 \\ \Phi_{t_1} \end{pmatrix} = \begin{pmatrix} F^* \\ F^* D_{t_1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ W^2 & W^3 & 1 & W & W^2 \\ 1 & W^2 & 1 & W^2 & 1 \\ W^2 & W & 1 & W^3 & W^2 \\ W_{t_1}^{-2} & W_{t_1}^{-1} & 1 & W_{t_1} & W_{t_1}^2 \\ W_{t_1}^{-2} W^2 & W_{t_1}^{-1} W^3 & 1 & W_{t_1} W & W_{t_1}^2 W^2 \\ W_{t_1}^{-2} W^2 & W_{t_1}^{-1} W^2 & 1 & W W^2 & W_{t_1} W^2 \\ W_{t_1}^{-2} W^2 & W_{t_1}^{-1} W & 1 & W W^3 & W_{t_1} W^2 \end{pmatrix}
\] (15)
• Orthogonal of the vector $\phi^l_t$ to each other for any set of offset values $t$

• Having a different coefficients but same Fourier vector in overlapping frequency part

Fig. 6. Signal spectrum before (a) and after sampling (b), for $L=5$ and $N=4$. The base spectrum (-) and aliased spectrum (--) overlap for the first and last spectral component, corresponding to the first and fifth column in (15).
Theorem 3.1: For varying $t$, any vector $\phi^l_t$ (with $l \neq 0$) describes a trajectory in the $M$-dimensional subspace $\nu_l$ of $C^{MN}$:

$$\nu_l = \text{span}\left\{ A \otimes f_l \right\}$$

with

$$A = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & -1 & 1 & \cdots \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & -1
\end{pmatrix}$$

(16)

where $A$ is an $M \times M$ matrix.

$\otimes$ represents the Kronecker product.

- vector $\phi^{l+iN}_t$
  - Corresponding to overlapping spectrum coefficients
  - Belong to the same space $\nu_l$
• Vector $\phi_t^k$
  
  – Belong to the orthogonal subspace $\nu_k \perp \nu_l$

  – Proof: The trajectory of $\phi_t^k$ as function of $t$ is in $\nu_l$ iff we can write any arbitrary $\phi_t^k$ as a linear combination of the columns of $A \otimes f_l$

• Linear equations

• Unique solution when rank is full

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & -1 & 1 \\
\vdots & \ddots & \vdots \\
1 & 1 & \cdots & -1
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{M-1}
\end{pmatrix} =
\begin{pmatrix}
1 \\
W_{t_1}^l \\
\vdots \\
W_{t_{M-1}}^l
\end{pmatrix}
\] (17)

• Orthogonality of the Fourier basis

\[
\langle A^i \otimes f_k , A^j \otimes f_l \rangle = \sum_{n=0}^{M-1} (\pm 1) \langle f_k , f_l \rangle = 0 \quad \text{if } k \neq l
\] (18)
- Projection into the subspace

\[ y^{(l)} = P_{V_{l}} y \]  \hspace{1cm} (19)

Fig. 7. Illustration of trajectory of the span of two columns \( \phi_{l}^{k} \) (the planes in this drawing) in a three-dimension space \( V_{l} \) for different offset value \((0, t, \text{and } t')\). Only for the correct offset \( t \), the vector \( y^{(l)} \) belongs to the space spanned by the two columns.
- **example 3.2 : looking back at example 3.1**
  - *Projection of two aliasing column*
    - Mapping into the same subspace
    - No information about $t_1$
  - *Projection of no aliasing column*
    - Coincided for the correct value of $t_1$
  - *One dimension subspace ($\nu_0$)*
    - No information about $t_1$
Fig. 8. Difference between the sample vector projected onto the spaces $v_n$ and the corresponding columns of $\Phi_t$. The space $v_2$ contains two vectors, and $\|y^{(n)} - \hat{y}^{(n)}\|^2$ is zero for any value of $t$. Similarly, $v_0$ does not depend on $t$ and thus it gives no information about the offset either. Both $v_{-1}$ and $v_1$ clearly indicate the correct value of $t$. 
Solution using Matrix Rank

◆ Method
  – Finding a relative offsets
    • Discrete Fourier transform of samples
      \[ Y_m = \frac{1}{N} F_N F^* D_{tm} \alpha \]  
      (20)
      where \( F_N \) is a square \( N \times N \) DFT matrix.
      \( F^* \) is the \( N \times L \) IDFT matrix.
    • Split of the Fourier coefficient vector \( \alpha \) in blocks \( \alpha_i \) of length N
      \[ Y_m = D'_{tm} \sum_{i=[-(S-1)/2]}^{[(S-1)/2]} W_{tm}^{iN} \alpha_i \]  
      (21)
      where The vectors \( \alpha_i \) represents the overlapping parts of the Fourier spectrum.
      and There are \( S = \lceil L / N \rceil \) overlapping parts.
- Linear combination of the S parts of the Fourier spectrum $\alpha_i$.
  - Belong to the S-dimensional subspace $\text{span}\left(\left\{\alpha_i\right\}_{i=\left\lceil -(S-1)/2 \right\rceil}^{\left\lceil (S-1)/2 \right\rceil}\right)$ of each vector $D_{t_m}^{-1}Y_m$

$$D_{t_m}^{-1}Y_m = \sum_{i=\left\lceil -(S-1)/2 \right\rceil}^{\left\lceil (S-1)/2 \right\rceil} W_{t_m}^i \alpha_i$$  \hspace{1cm} (22)

- Rank of the matrix containing all the sets of samples.
  - Having more than S sets of samples ($M>S$), and ($N\geq S$)

$$\text{rank} \left( Y_t^D \right) = S$$

with $Y_t^D = \left( Y_0 \ D_{t_1}^{-1}Y_1 \ \cdots \ D_{t_{M-1}}^{-1}Y_{M-1} \right)$  \hspace{1cm} (23)
- Founding a correct values of the offset $t$
  - Becoming $S$ of the rank of matrix
  - Not correct value in $\text{rank}(\mathbf{Y}_t^D) > S$

---

**Fig. 9.** Signal reconstruction algorithm using the matrix rank method from the section IV. The estimate for the offsets $\hat{t}$ and the corresponding sample matrix $\mathbf{Y}_t^D$ are updated iteratively, Once the estimate is good enough, the signal parameters $\alpha$ are reconstructed.
– **Discussion**

• Least number of sample set
  \[
  M = S + 1 = \left\lceil \frac{L}{N} \right\rceil + 1
  \]

• Additional samples set
  – Robustness in estimation of offset
  – Increasing complexity of the estimation

• Solving the problem
  – S+1 singular value decomposition

\[
\min_{\hat{\sigma}} \sigma_{S+1} \left( Y_i^D \right)
\]

where \( \sigma_{S+1}(A) \) stand for computing the S+1th singular value of the matrix A.
– Example 4.1
  • Set condition 1
    – Bandlimited signal
    – L=81, M=2, N=90, t=(0 0.6)

Fig. 10. Example of the objective function in (24). A signal with 81 unknown Fourier coefficients is sampled with two aliased sets of 90 samples. The exact offset is $t_1=0.6$. Next to the global minimum, it also contains many local minima.
Set condition 2
- Bandlimited and aliased signal
- $L=81$, $M=3$, $N=41$, $t=(0 \ 0.2 \ 0.6)$

Fig. 11. Example of the objective function in (24). A signal with 81 unknown Fourier coefficients is sampled with three aliased sets of 41 samples. The exact offsets are $t_1=0.2$ and $t_2=0.6$. (a) Two-dimensional objective function has many local minima. (b) A slice of the error surfaces for $t_1=0.2$ is shown for the different singular values. The three singular values $\sigma_1$, $\sigma_2$ and $\sigma_3$ do not vary much, except for $\sigma_3$ at the correct offset value.
Solution using Projections

◆ Method

– Finding a relative offsets
  • Projections onto a subspace basis
    – Constructed in the correct values for the offset vector

\[
\begin{align*}
\hat{y} &= P_{\Phi_i} y = \Phi_i \left( \Phi_i^* \Phi_i \right)^{-1} \Phi_i^* y = y \quad \text{for} \quad \hat{t} = t \\
\hat{y} &= P_{\Phi_i} y \quad \text{for} \quad \hat{t} \neq t
\end{align*}
\]  

(25)

where \( P_{\Phi_i} \) is the projection operator.

\( \text{span}(\Phi_i) \) is the subspace of \( y \).

• Optimization for the correct offset

\[
\min_i \left\| y - \hat{y} \right\|^2
\]

(26)
Fig. 12. Signal reconstruction algorithm using the projection method from the section V. The estimate for the offsets $\hat{t}$ is updated iteratively. Once the estimate is good enough, the signal parameters $\alpha$ are reconstructed.

– Discussion
  
  • Number of samples
    – At least as many samples as the total number of unknowns
    – $MN > L + M - 1$
– Example 5.1
  • Set condition
    – Equal to example 5.1

**Fig. 13.** Examples of the objective function in (26). Next to the global minimum, it also contains many local minima. (a) Two aliased sets of 90 samples, with 81 unknown coefficients. The exact offset is $t_1=0.6$. (b) Three aliased sets of 41 samples, with 81 unknown coefficients. The exact offsets are $t_1=0.2$ and $t_2=0.6$. Small values are represented by dark pixels.
Splitting the function into subdimensional space

\[ \|y - \hat{y}\|^2 = \sum_{n=0}^{N-1} \|y^{(n)} - \hat{y}^{(n)}\|^2 \]  

(27)

where \( y^{(n)} \) is the projection of \( y \) onto the subspace \( \mathcal{U}_n \).

\( \phi^{n+iN}_t \) is projection of \( y \) onto the vector.

Fig. 14. Example of the decomposition of the objective function into its different components belonging to orthogonal \( M \)-dimensional subspaces.
Practical issues

◆ Sampling kernel
  – No sampling kernel
    • Using diracs kernel
      – Not very realistic
      – Simplifying the analysis
  – Considering a sampling kernel
    • Canceling the effect of sampling kernel

◆ Images and higher-dimensional signals
  – Using one-dimensional signal in simulation
  – Applying a method to images (high-dimensional signal)
    • Two-dimensional signal processing
      – Increasing computational complexity
Minimization

- Finding the global minimum of the objective function
  - Gradient descent approach
  - High probability close to the smallest value in grid
  - Computation complexity
    - Choosing initial point with another algorithm

Heuristics approaches

- Hierarchical approaches (algorithm 6.1)
  - Assuming Fourier coefficient are not arbitrary Gaussian random variables
    - Having a constant decay
    - Using low frequency information
– Keeping the P best pairwise alignments (algorithm 6.2)
  • Searching a fixed number P of local minima
    – Obtaining the group of possible offset values
    – Independent pairwise alignment
    – Between first and mth set of samples
  • Global minimum
    – Neighborhood of the value of group that has possible offsets
Computational complexity of the different methods

- Computing offset
  - Number of times the error function has to be evaluated multiplied with the number of operation
- Complexity of the objective function evaluation
  - Matrix rank method

\[
C_{\text{rank}} = O((M - 1)N + M^2 N + M^3 + aM^2) \\
= O(M^3 + M^2 N)
\]  

where \( O((M - 1)N) \) is construction of \( Y^D_i \).
\( O(M^2 N) \) is matrix multiplication.
\( O(M^3 + aM^2) \) is an iterative power method operation.
– Projection method

\[ C_{\text{proj}} = O(MNL^2 + L^3) \]  \hspace{1cm} (29)

• Reduced complexity
  – Precomputed in storage space
    \[ C'_{\text{proj}} = O(M^2L^2) \]  \hspace{1cm} (30)

• Approximated overall complexity
  – Block decomposition
    \[ C_{\text{proj,F}} = O(MN + S^2L + ML) = O(S^2L) \]  \hspace{1cm} (31)
  – Standard algorithm (section VI-C)
    \[ O(N^{M-1} + E) = O(N^{M-1}) \]  \hspace{1cm} (32)
– Hierarchical method

\[ O((M - 1)N + 5^{M-1} + E) = O(MN + 5^{M-1}) \tag{33} \]

**Table. 1.** Computational complexity for the different methods on 1-D signals. The total complexity is obtained by multiplying the number of function evaluations with the complexity of a single function evaluation. The cost to obtain the initial estimate in Alg. 6.1 is denoted as \( C_{\text{init}} \)

<table>
<thead>
<tr>
<th>Method</th>
<th>( M^3 + M^2 N )</th>
<th>( MNL^2 + L^3 )</th>
<th>( S^2 L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>regular grid ( N^{M-1} ) eval.</td>
<td>( N^{M-1}(M^3 + M^2 N) )</td>
<td>( N^{M-1}(MN L^2 + L^3) )</td>
<td>( N^M S^2 L )</td>
</tr>
<tr>
<td>hier. method (Alg. 6.1) ( E ) eval. + ( C_{\text{init}} )</td>
<td>( (M^3 + M^2 N)E + C_{\text{init}} )</td>
<td>( (MN L^2 + L^3)E + C_{\text{init}} )</td>
<td>( S^2 LE + C_{\text{init}} )</td>
</tr>
<tr>
<td>pairw. al. (Alg. 6.2) ( MN + 5^M ) eval.</td>
<td>( (MN + 5^{M-1}) (M^3 + M^2 N) )</td>
<td>( (MN + 5^{M-1})(MN L^2 + L^3) )</td>
<td>( (MN + 5^{M-1})S^2 L )</td>
</tr>
</tbody>
</table>
Results

Tested and compared in a number of simulations

- Assumption
  - One-dimensional signal
  - Bandlimited signal

- Test 1
  - Different amount of additive white Gaussian noise
    - Offset estimation error to SNR
    - Success rate to SNR
      » Relative number of simulation in which the error on the registration is smaller than 0.001
Fig. 15. Results of the different algorithms as a function of the signal-to-noise ratio (SNR) for white Gaussian noise signals. The matrix rank algorithm from section IV performs slightly better than the projection algorithm from section V. Both algorithms perform clearly better than the algorithms that do not use the aliasing for registration (method from [4] and from [31]). Parameter values of $M = 3$, $L = 81$, and $N = 41$ were used. (a) Offset estimation error as a function of the SNR of the sampled signals. An offset error of 1 corresponds to a shift over the entire signal period. (b) Success rate of the registration and reconstruction as a function of the SNR.
– Test 2

- Pink noise signal
  - Constant decay of coefficient

Fig. 16. Results of the different algorithms as a function of the SNR for pink Gaussian noise signals (with $1/\omega$ behavior). Parameter values of $M = 3$, $L = 81$, and $N = 41$ were used. (a) Offset estimation error as a function of the SNR of the sampled signals. An offset error of 1 corresponds to a shift over the entire signal period. The algorithm from [31] has lower average absolute error than the rank based algorithms presented here. They still outperform the algorithm from [4], however. (b) Success rate of the registration and reconstruction as a function of the SNR. Both algorithms presented here outperform the other algorithms and compute the offsets up to a precision of $10^{-3}$ in a large proportion of the simulations.
Test 3

- Function of the number of samples
  - Proper number of samples in each set
    » 28 < N < 41

![Fig. 17. Results of the different algorithms as a function of the number of samples N. Parameter values of M = 3 and L = 81 were used in all simulations, and for N, value from 25 to 45 were used. No noise was added in this setup. The algorithms from section V performs better than the algorithm form section IV if the number of samples per set N is not sufficiently high. (a) Offset estimation error as a function of the number of the sample per set N. An offset error of 1 corresponds to a shift over the entire signal period. (b) Success rate of the registration and reconstruction as a function of the number of the sample per set N.](image)
Applying 2D signal (image)
– Double resolution enhancement

Fig. 18. Results of the different algorithms on images. (a) One of the five 32 x 32 images used as input for Algorithm 6.2, with the rank-based method from section IV. (b) Reconstructed 63 x 63 image from the images from (a). (c) One of the five 8 x 8 images used as input for Algorithm 6.1, with the projection-based method from section V. (d) Reconstructed 15 x 15 image from the image from (c).
Conclusion

- Proposed two mathematical methods
  - Rank of the modified sample matrix
  - Projection onto subspaces

- Limitation of the methods
  - Computational complexity
    - Not real-time reconstruction
    - Applying method to very high resolution
      - Satellite imaging