Oblique lattice systems and its application to design halftone masks

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Abstract

- Proposal of this papers
  - Integral lattice-based halftone masks by the OLS
- Oblique lattice system (OLS)
  - Good mathematical treatments of lattice
  - Implement
    - To realize the local and global numberings concept
    - To know conditions of halftone mask
  - Realization of mask size, resolution, line angle, etc.
Introduction

- Holladay’s method
  - Deal with the lattice structures of clustered dot mask
  - 4 Flaws
    (1) Difficulty to realize the required line angle
      - Supercell technique
    (2) Low resolution or low total tones
      - Multicenter dot technique
    (3) Noise or irregularity of dot shapes and textured patterns
      - Blue noise interpolation technique
    (4) Difficulty in realizing the required mask size
      - No appreciate approach
Definition 2.1

- Two parallel lines in rectangular
  \[ y_1hx - x_1wy = wzh_1(z_1 \in Z) \]  \hspace{1cm} (1)
  \[ x_2hx + y_2wy = wzh_2(z_2 \in Z) \]  \hspace{1cm} (2)

  - \( R, Z, N \) are real, integral and natural numbers
  - \( w, h \in R \) with \( w, h > 0 \)
  - \( x_1, x_2, y_1, y_2 \in Z \) with \( x_1x_2 + y_1y_2 \neq 0 \)
  - Let \( \Lambda = |x_1x_2 + y_1y_2| \)

- Expression
  - \( \Lambda[(w,h),(x_1,y_1),(x_2,y_2)] \) or \( \Lambda \)
◆ Geometric explanation of OLS

- Procedure with the case, $|x_1| \geq |y_1| \geq 0$
  - Diving both side with $|x_1|$ pieces
  - Setting both $|x_1| - 1$ dividing points
  - Connecting lower left corner point and the $|y_1|$th dividing point from the upper corner point with $x_1 y_1 \geq 0$
  - Placing all lines from left to right points

- Procedure with the case, $|y_1| \geq |x_1| \geq 0$
  - Interchange $|x_1|$ and $|y_1|$
  - Interchange left and right with lover and upper edges
– Example of OLS $\Lambda[(128,128), (2,8), (2,12)]$

Fig 1. OLS $\Lambda[(128,128), (2,8), (2,12)]$
◆ Properties of OLS

1. \( \theta_1 = \tan^{-1}\left(\frac{h v_1}{w x_1}\right) \), \( \theta_2 = -\tan^{-1}\left(\frac{h x_2}{w y_2}\right) \).

2. \( \theta = \begin{cases} 
\tan^{-1}\left[\frac{w h (x_1 x_2 + y_1 y_2)}{w x_1 y_2 - h x_2 y_1}\right] & w x_1 y_2 - h x_2 y_1 \neq 0, \\
\pi/2 & w x_1 y_2 - h x_2 y_1 = 0.
\end{cases} \)

3. \( l_1 = \frac{w h}{(w^2 x_1^2 + h^2 y_1^2)^{1/2}} \), \( l_2 = \frac{w h}{(h^2 x_2^2 + w^2 y_2^2)^{1/2}} \).

4. Tile by a periodic square

5. Expression of position \((x, y)\)

\[ a_1 v_1 + a_2 v_2 \]

where \( a_1 \in R, a_2 \in Z, \) or \( a_1 \in Z, a_2 \in R \), and

\[ v_1 = \left(\frac{w x_1}{d n}, \frac{h y_1}{d n}\right), v_2 = \left(\frac{w y_2}{d n}, -\frac{h x_2}{d n}\right). \]
6. Total number of lattice : \( dn \)

7. All lattice points of \( \Lambda \)
\[
a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + (w, h) \left[ \text{mod}(w, h) \right] \begin{cases} 
    a_1 = 0, \ldots, dn / d - 1 \\
    a_2 = 0, \ldots, d - 1
\end{cases}
\]

where \( \text{mod}(w, h) \) means taking each \( x, y \) component’s modulo by \( w, h \) in turn, and \( d \in \mathbb{N} \) is the greatest common divisor of \( |x| \) and \( |y| \).

8. Equal OLSs
\[
\Lambda[(w, h), (\pm x_1, \pm y_1), (\pm x_2, \pm y_2)], \\
\Lambda[(w, h), (\pm y_1, \mp x_1), (\pm y_2, \mp x_2)]
\]

9. Having same lattices points of OLSs
\[
\Lambda[(w, h), (ix_1 + jy_2, -jx_2 + iy_1), (kx_2 + ly_1, -lx_1 + ky_2)],
\]

where each \( i, j, k, \) and \( l \) is in \( \mathbb{Z} \) with \( |ik + jl| = 1 \).
10. Same lattice structure

\[ \Lambda[(w 2^{p_1}, h 2^{p_2}), (x_1 2^{p_2}, y_1 2^{p_1}), (x_2 2^{p_1}, y_2 2^{p_2})], \]

where \( p_1, p_2 \in \mathbb{Z} \) with each \( x_1 2^{p_2}, y_1 2^{p_1}, x_2 2^{p_1}, y_2 2^{p_2} \in \mathbb{Z} \).

◆ Definition 2.3

– Subdivided OLS of \( \Lambda \):

\[ \Lambda[(w, h), (x_1 - y_2, x_2 + y_1), (x_2 - y_1, x_1 + y_2)] \]

– Superdivided OLS of \( \Lambda \):

\[ \Lambda[(w, h), ((x_1 - y_2) / 2, (x_2 + y_1) / 2), ((x_2 - y_1)2, (x_1 + y_2) / 2)] \]

iif four fractional numbers are all integers.

Fig 3. OLS \( \Lambda[(w, w), (4,1), (5,2)] \) and its subdivision and superdivision
Proposition 2.4

- $n$ (natural number) times superdivided OLS of $\Lambda$ with even $n$
  \[
  \Lambda[(w, h), (x_1/2^{n/2}, y_1/2^{n/2}), (x_2/2^{n/2}, y_2/2^{n/2})]
  \]
  if each $x_1$, $y_1$, $x_2$, and $y_2$ can be divided by $2^{n/2}$ into an integral number.

- $n$ times superdivided OLS of with odd $n$
  \[
  \Lambda[(w, h), ((x_1 - y_2)/2^{(n+1)/2}, (x_2 + y_1)/2^{(n+1)/2}), ((x_2 - y_1)/2^{(n+1)/2}, (x_1 + y_2)/2^{(n+1)/2})]
  \]
  if each $x_1 - y_2$, $x_2 + y_1$, $x_2 - y_1$, and $x_1 + y_2$ can be divided by $2^{(n+1)/2}$ into an integral number.
**Design of halftone masks by OLSs**

- Uniform balanced numbering concept and its decomposition
  - Use fixed mask size \((w \times h)\) to overcome flaw(4)
    - Apply common conception to all halftone masks
  - Common concept
    - “the consecutive numbering 0,\ldots,wh – 1 so that the numbering is done by preserving uniform balance in whole of the mask of size \(w \times h\)
    - Determine the number of tones of halftone masks \((0,\ldots,\text{cols})\) from the consecutive numbers \((0,\ldots,wh – 1)\)
– Equation of halftone mask from common concept

\[ p_c[u(i)] = \begin{cases} i = 0, \ldots, wh-1 \\
\end{cases} \]

where \( u(i) \) means the uniquely determined pixel in \( w \times h \), and \( p_c \) gives the consecutive numbering value 0,..., \( wh-1 \) with uniform balance

– Decomposition

\[ p_c[u(i)] = p_c[u(j,k)] = \begin{cases} j = 0, \ldots, dn-1, \ k = 0,\ldots, cl-1 \\
\end{cases} \]

where \( u(j,k) \) is uniquely redetermined for \( j \) and \( k \), \( dn \) is the number of clusters in the mask, and \( cl(j) = wh / dn \) is the number of pixels in the \( j \)'th cluster with \( \sum_{j=0}^{dn-1} cl(j) = wh \)

• Core pixels of clusters : \( u(j,0) \)
• Pixels in the \( j \)'th cluster : \( u(j,k) \)
• lattice-based masks and stochastic clustered dot type masks from the numbering
– Perfect form of decomposition

• Place each core pixel on an integral lattice point

\[ p_c[u(i)] = p_c[u(j,0)] + p_c[u(0,k)] \quad j = 0, \ldots, dn - 1, \quad k = 0, \ldots, cl - 1 \]  

where \( p_l \) gives the local numbering value \( 0, \ldots, \text{wh} / dn - 1 \), and \( p_g \) gives the global numbering value \( 0, \ldots, dn - 1 \) with uniform balance.

• Regular on each cluster by local numbering and irregular on lattice pixels by global numbering
◆ Design algorithm by integral OLSs with the local and global numberings concept
  – Solution of the 4 flaws
    • Many combination of angle in the OLS
    • Independence of the resolution and the total tone from the consecutive numbering
    • Global numbering with stochastic property to prevent regular textured pattern
    • Possibility of the resolution size at first
– Algorithm A (the consecutive numbering algorithm)

1. Set $w \times h$ mask and the number of lattices $dn$ corresponding the lattice points

2. Set the cluster of pixels by the same shape which consist of $wh/dn$ pixels

3. Local numbering for $p_l[u(0,k)]$

4. Global numbering for $p_g[u(j,0)]$

5. Set the consecutive numbering for $p_c[u(j,k)]$

6. Arrange the number $(0, ..., wh - 1)$ to the total tone number $(0, ..., cols - 1)$ evenly
– Algorithm B (constraint of stochastic property)

• \( w \times h \) mask size for \( d_w \times d_h \) dpi

• Let \( P \) be a selectable (lattice points) and \( Q \) be a selected pixel set

• Let \( r \) be a real number: \( r < \frac{1}{2} \frac{w}{d_w}, \quad r < \frac{1}{2} \frac{h}{d_h} \)

• Let \( f : R \rightarrow R \) be function to measure the watching pixel’s density with the range \([1,0]\) and \( f(x) = 0(x > 1) \)

\[
\sum_{x \in Q} f\left[ \frac{d(x, y)}{r} \right]
\]

where \( d(x, y) \) be a distance from \( y \in P \) to \( x \in Q \)

\[
f(x) = \begin{cases} 
(2/3 - x + 1/3x^3)^2 & x \leq 1 \\
0 & x > 1 
\end{cases}
\]
• Select pixels from $P$ with adding to $Q$ in order that a pixel with lower density is selected faster
Control of the global numbering

- Use superdivided OLS for stochastic irregular patterns and regular textured patterns
- Let $v_1(i)$ and $v_2(i)$ be $i$ times superdivided for under $n$ times superdivided $1 \leq i \leq n$
- Procedure of re-sort $u(j,0)$
  1. Let $u(j,0)$ for $j = 0, \ldots, dn/2^n - 1$
  2. Let $i = n$
  3. Let $u(j,0) = u(j - dn/2^i,0) + [v_1(i) + v_2(i)]/2 + (w, h)[\text{mod}(w, h)]$
     for $j = dn/2^i, \ldots, dn/2^{i-1} - 1$
  4. $i = i - 1$ and repeat (2)-(4) until $i \geq 1$
- Result equation
  $$p_g[u(j,0)] = p_g[u(j \mod dn/2^n,0)] + (dn/2^n) \text{int}[j/(dn/2^n)]$$
  where $\text{int}$ means the round down operation to integers.
Homogeneous type masks

Definition 3.1 (Uniform and Homogeneous OLS)

• Assumption
  – $\Lambda$ be an integral OLS
  – $cols \in N$ be the total tone number and
  – $ds = \frac{wh}{cols} \in R$ be the number of pixels assigned to one tone

• Necessity for uniform OLS
  – Integral number of $ds$

• Necessity for homogeneous OLS
  – Existence of integral number, $p \geq 0$
  – Satisfaction $\frac{dn}{ds} = \frac{dn \cdot cols}{(wh)} = 2^p$
  – Possibility $p$ times superdivision of $\Lambda$

– Integral OLS $\supset$ uniform OLS $\supset$ homogeneous OLS
Corollary 3.2
- uniform OLS $\supseteq$ homogeneous OLS

Corollary 3.3 (Condition for homogeneous OLS)
- Necessity for homogeneous OLS $\Lambda$ with $cols = (wh / dn)2^p$
  1. $p$ is even and $x_1, x_2, y_1, y_2$ can be divided by $2^{p/2} = (dn / ds)^{1/2}$ into integral numbers.
  2. $p$ is odd and $x_1 - y_2, x_2 + y_1, x_2 - y_1$, and $x_1 + y_2$ can be divided by $2^{(p+1)/2} = (2dn / ds)^{1/2}$ into integral numbers

- Example
  - $\Lambda[(64,64),(16,8),(12,8)]$ with $cols = 256 \rightarrow ds = 16, p = 4$
  - $\Lambda[(32,32),(16,8),(12,8)]$ with $cols = 256 \rightarrow ds = 4, p = 6$
  - $\Lambda[(32,32),(16,8),(12,8)]$ with $cols = 200 \rightarrow ds = 5.12$
Corollary 4.1

1. Line angles \( \tan^{-1}\left(\frac{d_w h y_1}{d_h w x_1}\right) \), \( -\tan^{-1}\left(\frac{d_w h x_2}{d_h w y_2}\right) \).

2. Narrower angle between two line

\[
\begin{cases}
\tan^{-1}\left[ \frac{wh(x_1 x_2 + y_1 y_2)}{d_h w x_1 y_2 - d_w h x_1 y_2} \right] & \text{if } d_h w x_1 y_2 - d_w h x_1 y_2 \neq 0 \\pi/2 & \text{if } d_h w x_1 y_2 - d_w h x_1 y_2 = 0
\end{cases}
\]

3. Line resolution [lines/inch (=lpi)]

\[\left(\frac{d_h^2 w^2 x_1^2 + d_w^2 h^2 y_1^2}{wh}\right)^{1/2}, \left(\frac{d_h^2 w^2 x_2^2 + d_w^2 h^2 y_2^2}{wh}\right)^{1/2}\]

4. Corresponding resolution (lines/in.)

\[\sqrt{\frac{d_h d_w |x_1 x_2 + y_1 y_2|}{wh}}\]
Example 4.2

- 256 tone mask of $64 \times 64$ pixels for 600 dpi from OLS $\Lambda[(64,64), (16,8), (12,8)]$

- Result of equation before presented
  1. Mask size: $64 \times 64$ pixels
  2. The numbers of lattice points of $\Lambda: dn = 256$
  3. the basis of $\Lambda: \{(4,2), (2,-3)\}$
  4. Two line angles: $\tan^{-1}(1/2), -\tan^{-1}(3/2)$
  5. Two line resolutions: 168, 135 lpi
  6. Corresponding resolution: 150 lpi
  7. The number of total tones: $cols = 256$
  8. The amount of pixels for one tone: $ds = 16$
– Procedure

1. set the size of mask

2. Choose a cluster of pixels and local numbering $p_i$ from 0 to $wh/dn−1=15$

(a) line-shaped growth

(b) dot-shaped growth

Fig 4. Lattice pixels of a 64 x 64 size mask assigned by $\Lambda[(64,64),(16,8),(12,8)]$

Fig 5. Samples of local numbering
• Practical procedure of local numbering
  – Choose 50% \( \frac{wh}{2dn} - 1 = 7 \), and then number all

![Diagram with numbering examples]

3. Determine \( p_g \) from 0 to \( dn - 1 = 255 \) for each lattice pixel uniquely and superdivided 4 times
  - Original  \( \Lambda[(64,64),(16,8),(12,8)] \)
  - First superdivision  \( \Lambda[(64,64),(4,10),(2,12)] \)
  - Second superdivision  \( \Lambda[(64,64),(8,4),(6,4)] \)
  - Third superdivision  \( \Lambda[(64,64),(2,5),(1,6)] \)
  - Fourth superdivision  \( \Lambda[(64,64),(4,2),(3,2)] \)

Fig 6. Samples of line-shaped local numbering
Fig 7. Samples of the pair structure of superdivided lattice of $\Lambda[(64,64), (16,8), (12,8)]$

(a) 2nd  
(b) 1st  
(c) Paired 2nd

Fig 8. Samples of the global numbering by 16 for lattice point of OLS $\Lambda[(64,64), (16,8), (12,8)]$

(a) with all superdivided lattices  
(b) with no superdivided lattices  
(c) with 1st superdivided lattices between (a) and (b)
4. Let consecutive number be \( p_c = p_d n + p_g \), so that each pixel is numbered from 0 to 4095.

5. Rearrange the number from 0 to \( wh - 1 \), to from 0 to \( cols - 1 \) and renumber \( 256n \) to \( 256(n+1) \), for \( n = 0, ..., 255 \).
Fig 9. Mask patterns achieved by OLS \( \Lambda[(64,64),(16,8),(12,8)] \) (printed at 200 dpi)

(a) homogeneous type (with all superdivided lattices)

(b) stochastic type (with no superdivided lattices)
Example 4.3

- 256 tones mask of $16 \times 16$ pixels for 600 dpi
- Orthogonal type of OLSs
  - $\Lambda[(16,16),(8,8),(8,8)]: 424\text{ lpi}, 45 \text{ deg}$
  - $\Lambda[(16,16),(4,4),(4,4)]: 212\text{ lpi}, 45 \text{ deg}$
  - $\Lambda[(16,16),(2,2),(2,2)]: 106\text{ lpi}, 45 \text{ deg}$
  - $\Lambda[(16,16),(4,0),(4,0)]: 150\text{ lpi}, 0 \text{ deg}$
Fig 10. Mask patterns by two OLSs (printed at 200 dpi).

(a) homogeneous type by $\Lambda[(64,64),(4,4),(4,4)]$

(a) mask using 1st superdivided OLS of $\Lambda[(64,64),(16,8),(12,8)]$
Fig 11. Natural images samples (printed at 600 dpi)

(a) error diffusion
(b) void-and-cluster mask (64 x 64)
(c) homogeneous type mask by Λ[(64,64), (4,4), (4,4)]
(d) mask using 1st superdivided OLS of Λ[(64,64), (16,8), (12,8)]
Some remarks

◆ Relation to the Holladay’s construction
  – Get the Holladay’s construction from any integral OLSs
    • Calculate the min size OLS with the same basis
    • Construct a homogeneous type mask with total tone number of
      \[ cols = \frac{wh}{dn} \]

◆ Inverse computations of OLSs
  – To require integral OLSs that realize a required condition
  – Procedure
    • Set the mask size \((w, h)\) and by changing \(x_1, y_1, x_2, \) and \(y_2\)
    • Check OLS integral and corresponding resolution in \([r_i, r_h]\)
      where \(r_i\) and \(r_h\) are the required lover and upper resolution
    • Check two line resolution and line angle
Suitable conditions for stochastic global numbering

- Prevent the irregular periodic patterns
  - Select mask size large
  - Select not very high resolution
- Proper mask size
  - Larger than $32 \times 32$ and around 150 lpi resolution

For color printing

- Aim to avoid moire artifacts
- Method 1: Select some OLSs which have the same lattice pixels
  
  $C \Lambda[(64,64), (16,8), (12,8)]: 168$ lpi, 26.5 deg
  
  $M \Lambda[(64,64), (16,−8), (12,−8)]: 168$ lpi, −26.5 deg
  
  $Y \Lambda[(64,64), (16,8), (20,−8)]: 202$ lpi, 68.0 deg
  
  $K \Lambda[(64,64), (16,−8), (20,8)]: 202$ lpi, −68 deg
- Second method: Sliding all pixels
  
  \[
  C \Lambda[(64,64), (10,4), (12,2)]: 201\text{lp}i, 21\text{deg}
  
  M \Lambda[(64,64), (10, -4), (12, -2)]: 201\text{lp}i, -21\text{deg}
  
  Y \Lambda[(64,64), (8,8), (10,6)]: 218\text{lp}i, -60\text{deg}
  
  K \Lambda[(64,64), (8,-8), (10, -6)]: 218\text{lp}i, 60\text{deg}
  \]

- Additional benefits by OLSs
  
  - Combination of angel and resolution in mask size
  - Control the number of total tones (cols) with tone reproduction control (TRC) correction
Conclusion

◆ Proposal
  – Simple algorithm for halftone masks by OLS that realize the concept of uniform balanced numbering
  – Global numbering by superdivided lattices

◆ Properties
  – Imposing regularity of clusters by local numbering
  – Imposing irregularity of dot lighting by local numbering with stochastic property