Correspondence

Corrections to "On the Exact Maximum Likelihood Estimation of Gaussian Autoregressive Processes"

BRUNO CERNUSCHI-FRÍAS AND JOHN D. ROGERS

The following corrections should be made to the above correspondence.[1]

1) In equations where capital italic P appears, it should always read lower case italic p. The letter p has two clearly distinct meanings:

a) it either designates a probability density function,

b) or it refers to the order of the process.

2) In (2) it is x_p and not xp.

3) The right-hand side of (3) is

$$\frac{1}{(2\pi)^{p/2}|\Gamma|^{1/2}} \exp\left[-(1/2)\, x_p^T \Gamma^{-1} x_p\right] \cdot \frac{1}{(2\pi)^{(N+1-p)/2}\sigma^{N+1-p}} \cdot \exp\left(-(1/2\sigma^2)\sum_{k=p}^{N}(x_k - a^T x_k)^2\right).$$

4) Equation (6.1) is

$$p(X/\theta) = \frac{1}{(2\pi)^{(N+1)/2}|\Gamma|^{1/2}} \cdot \exp\left[-(1/2)\, x_p^T \Gamma^{-1} x_p\right] \cdot \exp\left\{-(1/2\sigma^2)\left[a^T R a + \sum_{k=p}^{N} x_k^2 - 2a^T \sum_{k=p}^{N} x_k\right]\right\}.$$

5) In item d) at the bottom of p. 923, the correct equation is

$$\hat{\sigma}^{2(j)} = \hat{r}_0^{(j)} - \hat{a}^{(j)T} r^{(j)}.$$

6) The first sentence in the last paragraph of Section II should be: ". . . the computation of $\hat{\sigma}^{2(j)}$ from (5) may give a negative value for $\hat{\sigma}^{2(j)}$ *or* $|\hat{\Gamma}^{(j)}|$; . . . ."

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[1] B. Cernuschi-Frías and J. D. Rogers, *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, pp. 922-924, June 1988.

A New Window and Comparison to Standard Windows

YEONG HO HA AND JOHN A. PEARCE

Abstract—**A new window is introduced. It is shown that the window $w(t) = 0.62 - 0.48\,|t| + 0.38 \cos 2\pi t$, $|t| \leq 1/2$ is of the group having similar but different tradeoff properties with the Bartlett, the Hanning, and the Hamming windows.**

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I. Introduction

In the numerical evaluation of Fourier integrals or in the spectral estimation, windows are used to reduce the leakage and spectral bias [1]-[3]. Several standard windows are used to optimize the requirements of a particular application in signal processing. These are the Bartlett, the Hamming, and the Hanning windows, etc. These windows have good overall properties and are each optimum in some specific sense.

The common properties of these windows can be summarized as follows.

1) They are real, even, nonnegative, and time limited.

2) Their Fourier transforms have mainlobe at the origin and sidelobes at both sides. These sidelobes are decaying with asymptotic attenuation of f^{-n} as $f \to \infty$ where n is an integer.

In the following section, a new window is described and compared to several standard windows in terms of parameters which are generally used in the evaluation of a window.

II. Derivation of the Window

We designate $w(t)$ a window function and let its Fourier transform be $W(f)$. If $w(t)$ is real, even, unity at the origin and time limited:

$$w(0) = \int_{-\infty}^{\infty} W(f)\, df = 1 \tag{1}$$

$$w(t) = 0 \quad \text{for } |t| > 1/2 \tag{2}$$

then the transform pair $w(t)$ and $W(f)$ form the window pair.

With the constraints of window, a new window can be derived from a weighted sum of the Bartlett and Hanning windows to reduce the sidelobes. From the spectra of the windows, the Bartlett and the Hanning windows have partly opposite signs in the sidelobes. This situation immediately suggests that a linear combination of Bartlett and Hanning windows can be used to reduce certain effects of the sidelobes, especially near sidelobes. This can be achieved without broadening the window's mainlobe frequency response, resulting in a reduction of spectral resolution. The outcome can be called the modified Bartlett-Hanning window.

The proposed window function can be written by

$$w(t) = \begin{cases} \xi(1 - 2|t|) + (1 - \xi)(0.5 + 0.5 \cos 2\pi t), \\ \qquad |t| \leq 1/2 \\ 0, \quad \text{elsewhere} \end{cases} \tag{3}$$

where $0 \leq \xi \leq 1$.

The Fourier transform of the proposed function is

$$W(f) = \xi \frac{\sin^2(\pi f/2)}{2(\pi f/2)^2} + (1 - \xi)\frac{\sin \pi f}{2\pi f(1 - f^2)}. \tag{4}$$

The aim of this proposed window is to reduce the effect of sidelobes, using

$$\int_2^{\infty} W(f)\, df = 0 \tag{5}$$

as a measure. Solving (5) gives ξ approximately equal to 0.24.

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Therefore, the complete form of the window function is given by

$$w(t) = 0.24(1 - 2|t|) + 0.76(0.5 + 0.5 \cos 2\pi t)$$
$$= 0.62 - 0.48\,|t| + 0.38 \cos 2\pi t, \qquad |t| \le 1/2 \qquad (6)$$

and

$$W(f) = 0.12 \frac{\sin^2(\pi f/2)}{(\pi f/2)^2} + 0.38 \frac{\sin(\pi f)}{\pi f(1 - f^2)}. \qquad (7)$$

Equations (6) and (7) satisfy the conditions required for the window function and they are the window pair. Fig. 1 shows the plots of the modified Bartlett-Hanning and its decibel amplitude response.

III. Comparison of the Windows

We shall now compare the proposed window to the existing standard windows [2] in terms of the parameters [4] that reflect the effects of resolution degradation due to the mainlobe, leakage due to the near sidelobes, and leakage due to the far sidelobes. From Fig. 1(b), the following parameters of $|W(f)/W(0)|$ are selected for comparison.

b: The frequency at which the mainlobe drops to the peak ripple value of the sidelobes.

a_1: The peak ripple value of the sidelobes.

a_2: The ripple value of the sidelobes at $f = 64$, with f normalized according to (2), the value at which one can readily presume that asymptotic behavior has been reached.

d: Asymptotic decay rate of the sidelobe envelope.

The smaller b is, the better the resolution of the estimates; the smaller a_1 is, the smaller the leakage through near sidelobes; the smaller a_2 and the larger d are, the smaller the leakage through far sidelobes.

To evaluate the proposed window further, the following parameters are additionally of interest [2], [5].

Energy:

$$I = \int_{-\infty}^{\infty} W^2(f)\, df = \int_{-1/2}^{1/2} w^2(t)\, dt = 0.365. \qquad (8)$$

Amplitude moment:

$$m = \int_{-\infty}^{\infty} f^2 W^2(f)\, df = -w''(0) = 15.0. \qquad (9)$$

Asymptotic attenuation for large f:

$$W(f) = 0.12 \frac{\sin^2(\pi f/2)}{(\pi f/2)^2} + 0.38 \frac{\sin(\pi f)}{\pi f(1 - f^2)}$$
$$\underset{f\to\infty}{\cong} \frac{0.24}{\pi^2 f^2}. \qquad (10)$$

Thus, $W(f)$ goes to zero as f^{-2} for $f \to \infty$.

Table I shows the comparison of various windows via these parameters.

As an actual example of the use of the window, an estimate of the power spectrum of the signal (obtained by computing the discrete Fourier transforms of a windowed version of the correlation function) is considered and compared to the estimates using other windows. To illustrate the example, a test problem is provided in which

$$x(n) = \cos(2\pi n/10). \qquad (11)$$

This sequence is sinusoid with frequency $f_s/10$ where f_s is the sampling frequency. Fig. 2 shows the plots of the log power spectrums for this example.

A comparison based on Table I and Fig. 2 shows that the Bartlett, the Hanning, the Hamming, and the modified Bartlett-Hanning windows are of the group having similar properties. The proposed window is more effective than the Bartlett and the Hanning in the near sidelobes, and more effective than the Bartlett and the Hamming in the far sidelobes. No window is generally the best in all aspects, and one should select one according to the requirements of a particular application.

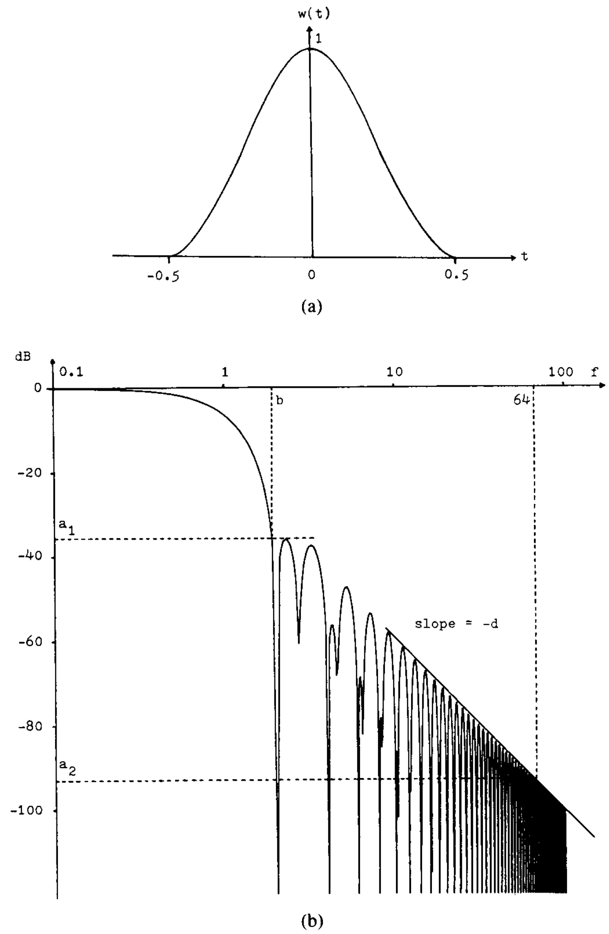

Fig. 1. Plot of the modified Bartlett-Hanning window. (a) Modified Bartlett-Hanning window. (b) Decibel amplitude response. Parameters b, a_1, a_2, and d are used to compare the window to others.

TABLE I
Comparison of Various Windows

Window	b	a_1	a_2	d	I	m	Asymptotic Attenuation
Modified Bartlett-Hanning	1.89	−36	−98	12	0.365	15.0	$0.24/\pi^2 f^2$
Rectangular	0.81	−13	−46	6	1.0	·	$1/\pi f$
Cosine-tip	1.35	−23	−84	12	0.5	9.87	$0.5/\pi f^2$
Bartlett	1.63	−26	−80	12	0.333	∞	$2/\pi^2 f^2$
Hanning	1.87	−32	−118	18	0.375	19.74	$0.5/\pi f^3$
Hamming	1.91	−43	−63	6	0.398	·	$0.16/2\pi f$
Papoulis	2.70	−46	−145	24	0.268	39.48	$2/\pi^2 f^4$
Blackman	2.82	−58	−126	18	0.305	32.37	$0.18/\pi f^3$
Parzen	3.25	−53	−136	24	0.269	48.0	$96/\pi^4 f^4$

References

[1] K. G. Beauchamp and C. K. Yuen, *Digital Methods for Signal Analysis.* London: George Allen & Unwin Ltd., 1979.

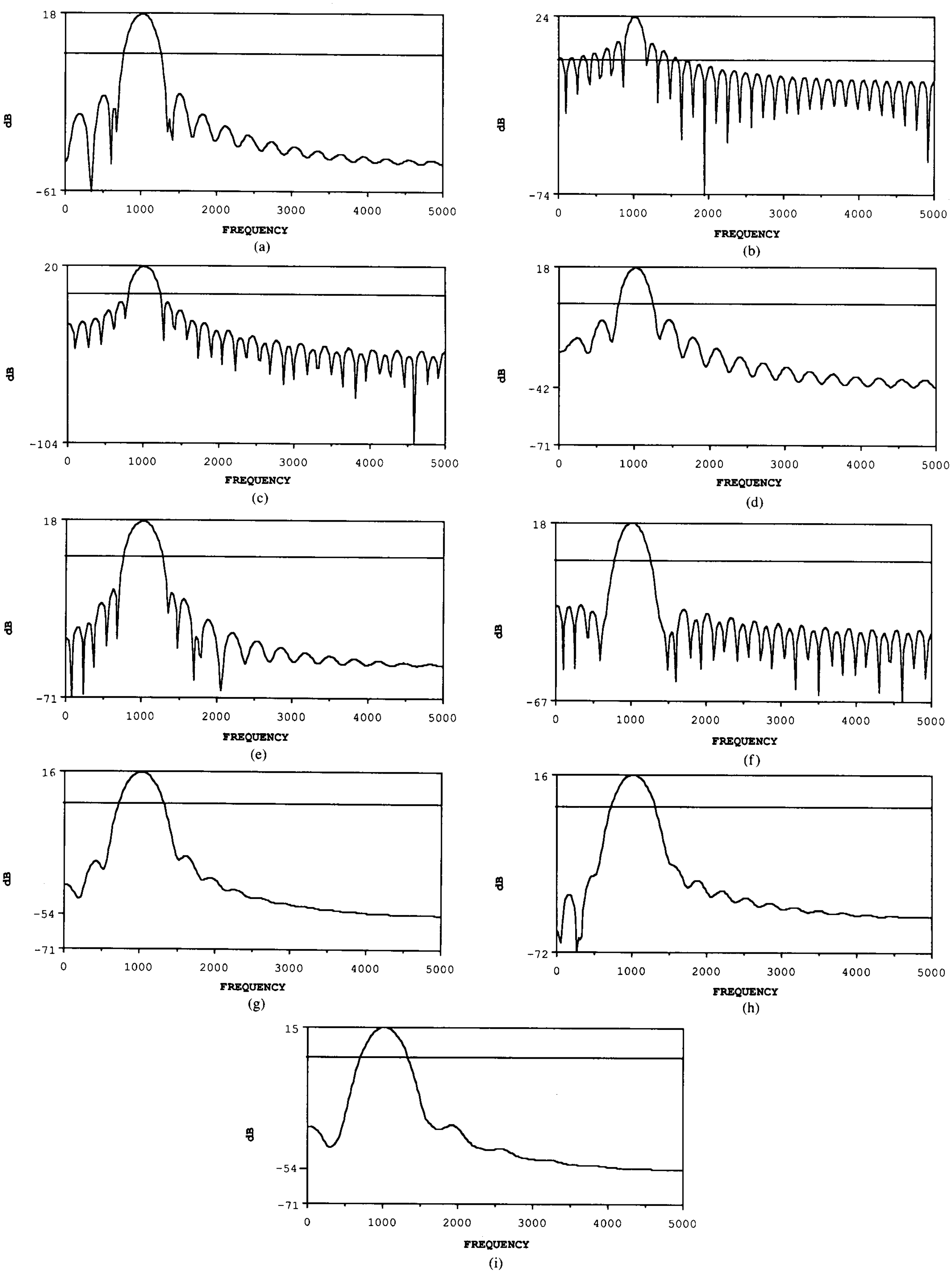

Fig. 2. Plots of the log power spectrum using various windows. Spectrum size = 128, total number of samples = 256, sampling frequency f = 10 000, number of correlation points used in estimating the power spectrum = 32, and FFT size = 512 are used as parameters. (a) Modified Bartlett-Hanning window. (b) Rectangular window. (c) Cosine-tip window. (d) Bartlett window. (e) Hanning window. (f) Hamming window. (g) Papoulis window. (h) Blackman window. (i) Parzen window.

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Improved Addition for the Logarithmic Number System

HARTMUT HENKEL

Abstract—**The logarithmic number system (LNS) offers a wide dynamic range with an independently choosable signal-to-noise ratio. To realize addition, voluminous lookup tables are needed. A method is proposed to reduce these tables by means of Chebyshev approximation with unequally spaced partition points.**

I. Introduction

In the LNS [2]–[14], a number X is expressed as $X = \pm r^e$ where r is the radix of the LNS and e is the exponent. The exponent e is composed of the following elements: $e = \pm e_{\alpha-1} \cdots e_1 e_0 \odot e_{-1} e_{-2} \cdots e_{-\beta}$ where $\odot$ denotes the radix point. The values $e_{\alpha-1} \cdots e_{-\beta}$ represent single bits. Therefore, every signed number X belonging to the permitted range of numbers is coded by a word of $N = 2 + \alpha + \beta$ bit.

Multiplication and division of numbers are simply accomplished by addition and subtraction of exponents e. These operations can be performed very fast and no error is introduced to the result.

For the addition of two numbers X and Y, the equivalence $X + Y = X \cdot (1 + Y/X)$ is very useful. This transform was first proposed by Gauss in 1812 (see [1]) for adding two numbers if their logarithms are known. By this transform, under the assumption that $X \geq Y$, the sum $Z = X + Y = r^{e(Z)}$ can be expressed as $e(Z) = e(X) + \Phi(A)$ where $\Phi(A) = \log_r(1 + r^{-A})$, with $A = e(X) - e(Y)$. An equivalent transform holds for subtraction. The function $\Phi(A)$ can be stored. For the addition, the address space of the memory is 2^N words, which tends to become very large if a small addition error is desired.

There exist some proposals to reduce the required address space [10], [13], [14]. Reference [13] makes use of the slope behavior of the function $\Phi(A)$. The slope m of $\Phi(A)$ is $m = -2^{-A}/(1 + 2^{-A}) < 2^{-A}$, which halves roughly with every incrementing of A by 1. If $\Phi(A)$ is mapped in PROM, usually table entries are optimized, so that for $A \approx 0$, neighbored entries differ by 1 LSB. Then one will find entries twice for $A \approx 1$, fourfold for $A \approx 2$, and so with increasing A more and more consecutive addresses are mapped to the same entry. The suggestion of [13] is to compress the address space by shifting the fractional part of A right n bits with n = integer (A) before addressing the lookup table. By doing this, the entropy of the PROM contents is increased and storing the complete table for $\Phi(A)$ needs only twice the address space as the table for $0 < A < 1$. For a relative addition error $f = 0.0001$ with the descripted compression scheme, 8192 words are requisite to store the whole function $\Phi(A)$. The proposal as follows makes use of interpolation methods by which the amount of memory can be reduced by a factor of 100 compared to [13].

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II. The Chebyshev Approximation Approach

To realize the function $\Phi(A)$ with a reduced table size, the maximum relative addition error F is introduced. For a positive error F,

$$\log_r((X + Y)(1 + F)) = \log_r(X + Y) + \log_r(1 + F)$$

is obtained and

$$\log_r((X + Y)(1 - F)) = \log_r(X + Y) + \log_r(1 - F)$$

for a negative error $-F$, respectively. If $F \ll 1$, it can be written

$$\log_r((X + Y)(1 - F)) \approx \log_r(X + Y) - \log_r(1 + F).$$

Thus, the addition error is equivalent to an additive uncertainty of the function $\Phi(A)$. The function $\Phi(A)$ is embedded in a tolerance field with the height $2\log_r(1 + F)$. The main point is to approximate the function $\Phi(A)$ within this tolerance field by a piecewise linear function $\Psi(A)$. A program was written to compute the optimum polygon by iteration, given the error F. The actual error of addition is a function of A. It is maximum negative at the partition points of the segments and maximum positive in the middle area of each segment. Table I shows the results for $r = 2$ and $F = 0.01$. In this example, four segments are sufficient to approximate the function $\Phi(A)$. Fig. 1 is a diagram of the function $\Phi(A)$ and the first two segments of the function $\Psi(A)$. For all values $A > -\log_r(F)$, $\Psi(A) = 0$ will not introduce any additional error. In this case, no addition is performed. The value $\Psi(A) = 0$ can be obtained by an early test of the value of A [10].

The number of segments (NOS) required for addition was computed as a function of the maximum relative error F and is shown in Table II. A good estimation for NOS is the function

$$\text{NOS} = \frac{2}{5 \cdot \sqrt{F}}.$$

The function $\Psi(A)$ is described in each interval by the values A_{start}, $\Psi(A)_{\text{start}}$, and slope_of_$\Psi(A)$. These values are stored in a memory, which is addressed by the actual segment number.

The distance between the partition points increases with increasing value of A, and so an additional memory is required to derive the actual segment number from the input A. If the partition points are slightly shifted, this memory can be replaced by a simple hardware logic. To variate the partition points means that a new function $\tilde{\Psi}(A)$ is created, which is somewhat suboptimum because a few more segments are necessary to perform the addition for a given errror F.

In real time processing, two steps are requisite to approximate the function $\Phi(A)$ by $\tilde{\Psi}(A)$.

1) Compute the number of the actual segment of $\tilde{\Psi}(A)$ from given A.

2) Approximate $\Phi(A)$ by a value of $\tilde{\Psi}(A)$, which is calculated by linear interpolation between the partition points of the actual segment.

III. Hardware Design Steps

A hardware logic transforms the input A into the segment number and the value $A - A_{\text{start}}$. This logic can be developed for any given F in a similar manner as shown in the following example for $r = 2$ and $F = 0.0001$.

At the beginning, the polygon $\Psi(A)$ is computed as an approx-

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